CCES – College of Computer Engineering and Sciences

Prince Sattam Bin Abdulaziz University

CS600: Advanced AI

First-Order Logic

Chapter 5 — Predicates, Quantifiers, Unification, and Resolution

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Artificial Intelligence Course

Chapter Outline | What we will cover

Learning Objectives

- See the **limitations** of propositional logic
- Master FOL **syntax**, **quantifiers**, and **semantics**
- Apply **unification**, substitution, and **Skolemization**
- Reason with **chaining** and **resolution**

Section 1

Why FOL & Syntax

The problem

"All mammals give birth to live babies; a cat is a mammal; therefore a cat does."

In propositional logic you would need a *separate rule for every object* — unwieldy or impossible for large domains.

FOL adds

- **Objects** — entities in the domain
- **Relations** — links between objects
- **Functions** — object $\rightarrow$ object
- **Quantifiers** — $\forall$ (all), $\exists$ (some)
- **Variables** — bind to any object

SQL, RDF/OWL, and Prolog are all built on FOL.

- **Constants**: JOHN, BEEF
- **Variables**: x, y, z
- **Functions**: $\text{Father}(x)$ — object $\rightarrow$ object
- **Predicates**: $E(x, y)$ — return truth values

Terms (denote objects)

constant ▪ variable ▪ function of terms: $\text{Father}(\text{Father}(x))$

Atomic formula

$P(t_1, \ldots, t_n)$ applies a predicate to terms.

$$F(\text{Spielberg}, \text{SPR}, \text{BestDir}) = \text{True}$$

Same connectives as before: $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, now over predicates.

Section 2

Quantifiers & Semantics

Universal $\forall$

"for all x" — a big **conjunction**.

$$\forall x\,[\mathrm{Mammal}(x) \to \mathrm{LiveBirth}(x)]$$

$\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge \cdots$

Existential $\exists$

"there exists x" — a big **disjunction**.

$$\exists x\,[N(x) \wedge E(x, \mathsf{Beef})]$$

$\exists x\, P(x) \equiv P(a) \vee P(b) \vee \cdots$

Common mistake

$\exists x\,[N(x) \to E(x, \mathsf{Beef})]$ is **wrong** for "some non-vegetarian eats beef" — it is true whenever any vegetarian exists. Use $\wedge$ with $\exists$.

Mixed order matters

$$\forall x \, \exists y \; E(x, y)$$

everyone eats *something* (maybe different).

$$\exists y \, \forall x \; E(x, y)$$

one food *everyone* eats — much stronger!

$$\forall x \, \exists y \;\not\equiv\; \exists y \, \forall x$$

De Morgan for quantifiers

$$\neg(\forall x \, \alpha) \equiv \exists x \, \neg\alpha$$
$$\neg(\exists x \, \alpha) \equiv \forall x \, \neg\alpha$$

Pushing $\neg$ through a quantifier **flips its type**.

Same-type quantifiers commute: $\forall x \forall y \equiv \forall y \forall x$.

An **interpretation** fixes:

- a non-empty **domain**
- each **constant** $\to$ an object
- each **function** $\to$ a mapping
- each **predicate** $\to$ truth values

$\forall x\, \alpha$ true iff α holds for *every* object; $\exists x\, \alpha$ iff for *some*.

Valid / Satisfiable

Valid: true in all models — $\forall x\,[P(x) \lor \neg P(x)]$.
Unsatisfiable: $\exists x\,[P(x) \land \neg P(x)]$.

Entailment

$KB \models \alpha \iff (KB \land \neg\alpha)$ unsatisfiable — the basis of **proof by contradiction**.

Section 3

Unification & Inference

Substitution $\{x/t\}$ replaces a variable with a term:

$$P(x, f(y))\{x/a,\ y/b\} = P(a, f(b))$$

MGU

The **most general unifier** commits to the fewest bindings — e.g. unify $P(x), P(y)$ with $\{x/y\}$, not $\{x/a, y/a\}$.

Unify	θ	
$P(x),\ P(a)$	$\{x/a\}$	✓
$P(x, f(y)),\ P(a, f(b))$	$\{x/a, y/b\}$	✓
$P(x, f(y)),\ P(y, f(a))$	$\{x/a, y/a\}$	✓
$P(x, x),\ P(a, b)$	—	×

Occur check forbids $x = f(x)$ (infinite terms).

Generalized Modus Ponens

$$\frac{P_1', \ldots, P_n',\ (P_1 \wedge \cdots \wedge P_n \to Q)}{Q\theta}$$

where θ unifies each P_i' with P_i.

Rule $\forall x, y\, [E(x,y) \wedge M(y) \to N(x)]$ with $E(\text{John}, \text{Beef})$, $M(\text{Beef})$ gives $N(\text{John})$.

Instantiation

- **Universal**: $\dfrac{\forall x\ \alpha}{\alpha\{x/t\}}$
- **Existential**: introduce a *new* Skolem constant

Skolemization | Eliminate ∃ for inference

Skolem constant

∃ not inside ∀:

$$\exists x\, P(x) \;\Rightarrow\; P(c)$$

c is a new constant.

Skolem function

∃ inside ∀:

$$\forall x\, \exists y\, E(x, y) \;\Rightarrow\; \forall x\, E(x, f(x))$$

the choice of y **depends on** x.

Nested example

$$\forall x\, \exists y\, [\forall z\, \exists w\, P(x, y, z, w)]$$

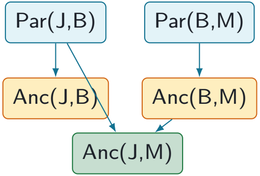
becomes

$$\forall x\, \forall z\, P(x,\ f(x),\ z,\ g(x, z))$$

- $f(x)$ depends on x
- $g(x, z)$ depends on x and z

Ancestor example

Rules: $\text{Parent}(x, y) \to \text{Anc}(x, y)$;
$\text{Parent}(x, z) \wedge \text{Anc}(z, y) \to \text{Anc}(x, y)$.
Facts: $\text{Parent}(\text{John}, \text{Bob})$, $\text{Parent}(\text{Bob}, \text{Mary})$.



	Forward	Backward
Driven by	Data	Goal
Search	Breadth	Depth
Space	Higher	Lower
Best for	Many queries	One query

Both sound & complete for definite clauses.

Section 4

Resolution & Summary

1. Eliminate $\rightarrow, \leftrightarrow$
2. Move $\neg$ inward (incl. quantifiers)
3. Standardize variables apart
4. **Skolemize** (remove $\exists$)
5. Drop universal quantifiers
6. Distribute $\vee$ over $\wedge$

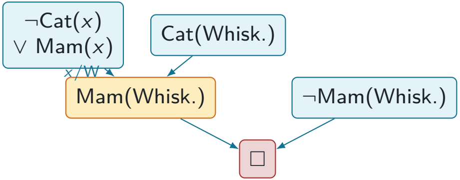
Example

$$\begin{aligned}
&\forall x\,[\mathrm{Cat}(x) \rightarrow \exists y\,\mathrm{Owns}(y,x)]\\
\Rightarrow\ &\forall x\,[\neg\mathrm{Cat}(x) \vee \exists y\,\mathrm{Owns}(y,x)]\\
\Rightarrow\ &\forall x\,[\neg\mathrm{Cat}(x) \vee \mathrm{Owns}(f(x),x)]\\
\Rightarrow\ &\neg\mathrm{Cat}(x) \vee \mathrm{Owns}(f(x),x)
\end{aligned}$$

Now a single CNF clause.

Resolve on complementary literals unified by θ:

$$\frac{\ell \vee A, \quad m \vee B}{(A \vee B)\theta}, \ \theta = \mathrm{UNIFY}(\ell, \neg m)$$



Empty clause $\square \Rightarrow KB \models \mathrm{Mammal}(\text{Whiskers})$.

Prove Whiskers is a mammal

(1) $\neg\mathrm{Cat}(x) \vee \mathrm{Mammal}(x)$
(2) $\mathrm{Cat}(\text{Whiskers})$
(3) $\neg\mathrm{Mammal}(\text{Whiskers})$ (negated query)

Propositional vs. First-Order Logic | More power, less decidability

Aspect	Propositional	First-Order
Objects / Relations / Functions	No	Yes
Quantifiers	No	Yes ($\forall, \exists$)
Expressiveness	Limited	High
Decidability	Decidable (NP-complete)	Semi-decidable

Semi-decidable: if $KB \models \alpha$, resolution will prove it; if not, it may run forever.

Represent

- Objects, relations, functions
- **Quantifiers** $\forall, \exists$ and variables
- Interpretations & entailment

Reason

Unification, Skolemization, chaining, and resolution.

Why it matters

Foundations of expert systems, the semantic web, databases (SQL), planning, and theorem proving.

Looking ahead

Ch. 6: from logic to **machine learning** — inductive reasoning.

Thank you!

Questions & Discussion

Next: Chapter 6 — Machine Learning